Set Theory and Mathematical Reasoning

Exercises

Page numbers refer to the full notes, `settheory.pdf`.

1 Sets and numbers

Exercise 1.1 (p. 6) For every set, there is at least one correspondence between it and itself.

Exercise 1.2 (p. 6) For a set with two elements there are precisely two correspondences between it and itself. For a set with three elements there are precisely six correspondences between it and itself.

Exercise 1.3 (p. 6) If there is a correspondence between A and B, and a correspondence between B and C, then there is a correspondence between A and C.

Exercise 1.4 (p. 6) If there is a correspondence between A and B, then there is a correspondence between B and A.

Exercise 1.5 (p. 7) Counting a set establishes a correspondence between it and the natural numbers below its cardinality.

Exercise 1.6 (p. 7) If you can count two sets, and get the same cardinal number, then there is a correspondence between the two sets.

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2 Well-orderings, countable vs uncountable infinities

Exercise 2.1 (p. 12) Show that any relation that is transitive (if x is before y and y is before z, then x is before z) and irreflexive (there is no x with x coming before x) must be antisymmetric (if x is before y then y is *not* before x).

Exercise 2.2 (p. 12) Show that the lexicographic ordering on ordered pairs of natural numbers is a well-ordering.

Exercise 2.3 (p. 13) What is $(\omega + 1) + (\omega + 3)$? What is $(\omega + 2) + (\omega + 2)$? What is $(\omega + 3) + (\omega + 3)$? What is $(\omega + 6) + \omega$? Which, if any, of these are equal?

Exercise 2.4 (p. 14) What is $2 \cdot (\omega + 3)$? What is $(\omega + 3) \cdot 2$? Is either of them equal to $\omega + \omega + 3 + 3$?

Exercise 2.5 (p. 17) Show that this same argument works for *any* set, not just the set of natural numbers. That is, let f be a function from S to $\mathcal{P}(S)$, let $D = \{x \in S \mid x \notin f(x)\}$, and show that D isn't in the range of f.

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3 Many countably infinite sets

Exercise 3.1 (p. 19) Any infinite subset of a countably infinite set is countably infinite.

Exercise 3.2 (p. 19) The union of two countably infinite sets is countably infinite.

Exercise 3.3 (p. 19) The union of countably many sets that are each countably infinite is countably infinite.

Exercise 3.4 (p. 19) The set of all finite sequences of natural numbers is countably infinite.

Exercise 3.5 (p. 20) The set of all grammatical sentences in English is countably infinite.

Exercise 3.6 (p. 20) The set of all syntactically correct computer programs is countably infinite.

Exercise 3.7 (p. 20) The set of all definitions of sets of natural numbers is countably infinite.

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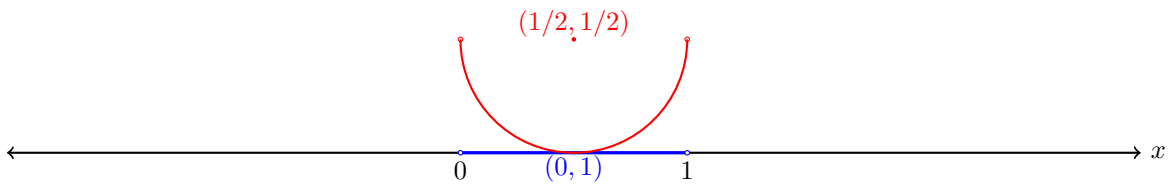
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4 Comparing the real numbers and the power set of the natural numbers

Exercise 4.1 (p. 22) Show, using just these axioms, and the ones for ordering and concatenation separately, that if $x_1 \circ y_1 = x_2 \circ y_2$, and $x_1 < x_2$, then $y_1 > y_2$. (Hint: If it weren't the case that $y_1 > y_2$, what would go wrong?)

Exercise 4.2 (p. 25) Consider the sequence of nested intervals $[n, n+1]$, $[n.d_1, n.d_1 + .1]$, $[n.d_1d_2, n.d_1d_2 + .01]$, ... This is the set of all quantities that share a given $(n+1)$th approximation. Use the Continuity principle to show that there must be *at least* one quantity contained in the intersection of all these intervals, and use the Archimedean axiom to show that there must be *at most* one quantity contained in the intersection of all these intervals. (Hint: If there were two distinct quantities contained in this intersection, see what wold happen if you count how many copies of their difference go into the original unit.)

Exercise 4.3 (p. 27) Show that there is a correspondence between the set $\{x \in \mathbb{R} \mid 0 < x < 1\}$ (all real numbers between 0 and 1, blue in the diagram below) and the set of points on the semicircle of radius $1/2$, centered at $(1/2, 1/2)$ (red in the diagram below), not including the endpoints of either the interval or the semicircle. Make sure to show that all four properties of a correspondence hold.



Exercise 4.4 (p. 28) Show that there is a correspondence between the set of points on the semicircle of radius $1/2$, centered at $(1/2, 1/2)$ (not including the endpoints) and the set of *all* real numbers. Make sure to show that all four properties of a correspondence hold.

Exercise 4.5 (p. 28) Show that there is a correspondence between the set of points in any particular finite interval of real numbers (not including endpoints) and the set of *all* real numbers.

Exercise 4.6 (p. 29) Using binary notation, show that there is a function from the real numbers in the interval from 0 to 1 (including endpoints) to $\mathcal{P}(\mathbb{N})$, the power set of the natural numbers, that is total, and one-to-one, but *not* onto. (Remember that in binary notation, $0.{}_2011111\cdots = 0.{}_21$ for the same reason that $0.09999\cdots = 0.1$ in decimal notation.)

Exercise 4.7 (p. 29) Using notation in any base greater than 2, show that there is a function from $\mathcal{P}(\mathbb{N})$ to the set of real numbers in the interval from 0 to 1 (including endpoints), that is total, and one-to-one, but *not* onto.

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5 The Axiom of Choice, the paradoxes

Exercise 5.1 (p. 33) Show (using the Axiom of Choice) that every set either has a correspondence with the set $\{0, \ldots, n\}$ for some natural number n, or it has a subset that has a correspondence with the natural numbers.

Exercise 5.2 (p. 34) Show (without using the Axiom of Choice, or anything complex) that if a set has a correspondence with a proper subset of itself, then there is a subset that has a correspondence with the natural numbers. (Hint: Let f be a correspondence between S and some proper subset of itself, and let 0 correspond to some element of S that is *not* in this subset.)

Exercise 5.3 (p. 34) Show (using the Well-Ordering Theorem, and some facts about ordinals from before) that for every set, there is a *least* ordinal that it has a correspondence with.

Exercise 5.4 (p. 34) Use the previous result to show that for every two sets, one of them has a correspondence with a subset of the other (this might not be a proper subset, if the sets have the same cardinality).

Exercise 5.5 (p. 35) Show that for any n, these (i.e., first player guesses white if they see an odd number of white hats; first player guesses black if they see an odd number of white hats) are the *only* two strategies that can guarantee that everyone but the first player guesses correctly.

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6 The formal language, and the "small" axioms: Extensionality, Pairing, Union, and Empty Set

Exercise 6.1 (p. 38) If $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$.

Exercise 6.2 (p. 39) Show that the objects $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$ all exist and are distinct (and thus that there are infinitely many distinct sets).

Exercise 6.3 (p. 40) $\langle x, y\rangle = \langle z, w\rangle$ iff $x = z$ and $y = w$.

Exercise 6.4 (p. 40) If $x \in X$ then $x \subseteq \bigcup X$.

Exercise 6.5 (p. 40) Show that for any objects $x_1, x_2, \ldots, x_n$, there is a set whose elements are all and only these objects. (This proof will need to work by induction, to show that it is true for any natural number n.)

Exercise 6.6 (p. 40) Find a set X and an $x \subseteq \bigcup X$ such that $x \notin X$.

Exercise 6.7 (p. 41) Using these definitions, and the results proved in the previous few paragraphs, show that each n is the set of all natural numbers strictly less than n. (This will be a proof by induction. The base case is to show that this is true when $n = 0$, where there is no natural number less than n, and the induction step involves showing that if it's true for n, then it's also true for $n + 1$.)

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7 The "medium" axioms: Power set, (Foundation), and Separation

Exercise 7.1 (p. 42) Find $\mathcal{P}(\emptyset)$, $\mathcal{P}(\mathcal{P}(\emptyset))$, and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ and write them explicitly using curly braces around "$\emptyset$".

Exercise 7.2 (p. 42) If $x \subseteq y$ then $\mathcal{P}(x) \subseteq \mathcal{P}(y)$.

Exercise 7.3 (p. 42) $x = \bigcup \mathcal{P}(x)$ and $x \subseteq \mathcal{P}(\bigcup x)$, but there exists a set x such that $x \neq \mathcal{P}(\bigcup x)$.

Exercise 7.4 (p. 43) Use the Separation axiom to prove that the empty set exists, using only the assumption that at least one set exists.

Exercise 7.5 (p. 43) Use the Separation axiom to prove that there is no set of all sets. (Hint: Use Russell's paradox, letting $\varphi(w)$ be "$\neg(w \in w)$".)

Exercise 7.6 (p. 44) If R is a relation from X to Y, and S is a relation from Y to Z, and R and S are both (functions/total/one-to-one/onto) then so is $R \circ S$ as a relation from X to Z.

Exercise 7.7 (p. 44) (Cantor's Theorem) Show that for any set X, there is no relation with domain X and range $\mathcal{P}(X)$ that is both a function and onto. (In this proof, you will need to be explicit about how, for any relation with domain X and range $\mathcal{P}(X)$ that is a function, you can use Separation to define a diagonal set, which is not part of the relation.)

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8 Infinity, formalized ordinals, and the well-ordering principle

Exercise 8.1 (p. 45) No natural number has a correspondence to a proper subset of itself. (Hint: The best way to prove this in the formal axiomatic context is to show that the set of natural numbers that don't have a correspondence to a proper subset is inductive.)

Exercise 8.2 (p. 45) ω does have a correspondence to a proper subset of itself.

Exercise 8.3 (p. 46) Use a similar sequence of operations to show that $S(S(0)) + (S(0) + S(0)) = S(S(S(S(0))))$ (that is, that $2 + (1 + 1) = 4$).

Exercise 8.4 (Commutativity of addition; p. 47) For all natural numbers m, n, $m + n = n + m$.

Exercise 8.5 (Associativity of addition; p. 47) For all natural numbers ℓ, m, n, $\ell + (m + n) = (\ell + m) + n$.

Exercise 8.6 (p. 48) Prove (by induction on n) that the combinatorial and recursive definitions of $m + n$ always agree.

Exercise 8.7 (p. 48) Prove (by induction on n) that the combinatorial definition of multiplication and the recursive definition agree for all m, n. (The base case will be very straightforward; for the induction step, there is a combinatorial definition of $m \cdot S(n)$, and there is a combinatorial definition of $m \cdot n + m$, and you will want to demonstrate a correspondence between the relevant sets in this combinatorial definition.)

Exercise 8.8 (p. 49) Show that for all natural numbers m, n, $m \cdot n = n \cdot m$.

Exercise 8.9 (p. 49) Show that for all natural numbers ℓ, m, n, $\ell \cdot (m \cdot n) = (\ell \cdot m) \cdot n$.

Exercise 8.10 (p. 49) Show, using the combinatorial definitions, that $m^2 = m \cdot m$. (This involves finding a correspondence between things like $f = \{\langle 0, k\rangle, \langle 1, \ell\rangle\}$ and ordered pairs of elements of m.)

Exercise 8.11 (p. 49) Show, using the combinatorial definition, that the power set of m has 2^m members. (This involves finding a correspondence between functions from m to $\{0, 1\}$ and subsets of m.)

Exercise 8.12 (p. 49) Show, by induction, that the recursive and combinatorial definitions of exponentiation agree.